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Quantum consistency of the superstring

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Abstract: Using arguments based on BRST cohomology, the pure spinor formalism for the superstring in an $AdS_5 \times S^5$ background is proven to be BRST invariant and conformally invariant at the quantum level to all orders in perturbation theory. Cohomology arguments are also used to prove the existence of an infinite set of non-local BRST-invariant charges at the quantum level.

Keywords: Superstrings and Heterotic Strings, AdS-CFT and dS-CFT Correspondence.

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Contents

1. Introduction

The superstring worldsheet action in an $AdS_5 \times S^5$ Ramond-Ramond background can be studied at the classical level using either the Green-Schwarz (GS) formalism [1] or the pure spinor formalism [2]. Since the $AdS_5 \times S^5$ Ramond-Ramond background is a solution of Type IIB supergravity, the worldsheet action is classically κ-invariant in the GS formalism [3] and is classically BRST-invariant in the pure spinor formalism [4]. In both these formalisms for the superstring, an infinite set of non-local classically conserved charges has been constructed which might be related to integrability [5, 6]. At the classical level, these non-local charges have been shown to be κ-invariant in the GS formalism and BRST-invariant in the pure spinor formalism [7].

For applications to the AdS-CFT conjecture, it is important to know if the worldsheet action and non-local charges remain κ-invariant or BRST-invariant after including quantum corrections. Because of quantization problems in the GS formalism, quantum κ-invariance is difficult to discuss, except perhaps near the plane-wave limit in which light-cone GS methods can be used [8]. However, since some isometries of the $AdS_5 \times S^5$ background are not manifest near the plane-wave limit, computations using this light-cone GS method appear quite complicated.

Using the pure spinor formalism for the superstring, there are no problems with quantization and one can easily discuss BRST invariance at the quantum level. In this paper, it will be proven using cohomology arguments that the BRST transformation of the quantum worldsheet effective action in an $AdS_5 \times S^5$ background can be cancelled by adding a local counterterm. The proof relies on the algebraic renormalization method of [9] in which trivial BRST cohomology at ghost-number +1 implies quantum BRST invariance to all orders in perturbation theory.

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Furthermore, it will be proven that after adding this local counterterm, the quantum worldsheet action is conformally invariant to all orders in perturbation theory. This proof uses a $U(2,2|4)$-invariant generalization of the worldsheet action and is similar to Witten's proof of quantum conformal invariance in [10] for the superstring in an $AdS_3 \times S^3$ Ramond-Ramond background. Note that it was previously shown by explicit computation that the worldsheet action in an $AdS_5 \times S^5$ background is conformally invariant at one-loop in α' [11]. And it was argued based on isometries that the $AdS_5 \times S^5$ background is not modified by higher-derivative corrections to the supergravity equations of motion [12].

In a recent paper, it was proven that whenever certain ghost-number 2 states are absent from the BRST cohomology, one can construct an infinite set of non-local BRST-invariant charges. It will be shown here that the ghost-number 2 cohomology is trivial in an $AdS_5 \times S^5$ background, implying the existence of an infinite set of non-local BRST-invariant charges at the quantum level.

At the classical level, these non-local BRST-invariant charges were shown in [7] to coincide with the classically conserved non-local charges found by Vallilo [6]. To explicitly construct the quantum non-local charges, one would first need to compute the quantum effective action. This computation is currently being done to one-loop order in collaboration with Brenno Carlini Vallilo [13], and some formulas in this paper have come from that collaboration. Although it is not obvious that quantum BRST invariance of the charges will automatically imply quantum conservation, it is reasonable to assume that BRST-invariant charges of zero ghost-number in the pure spinor formalism necessarily commute with the hamiltonian.[1]

In section 2 of this paper, the classical worldsheet action is reviewed using the pure spinor formalism for the superstring in an $AdS_5 \times S^5$ background. After adding appropriate local counterterms, the quantum worldsheet effective action is proven to be $SO(4,1) \times SO(5)$ gauge-invariant in section 3, BRST invariant in section 4, and conformally invariant in section 5. Finally, in section 6, an infinite set of non-local BRST-invariant currents are proven to exist at the quantum level.

2. Review of pure spinor formalism in $AdS_5 \times S^5$ background

In this section, the classical worldsheet action in an $AdS_5 \times S^5$ background is reviewed using the pure spinor formalism for the superstring. As in the Metsaev-Tseytlin GS action in an $AdS_5 \times S^5$ background [3], the action in the pure spinor formalism [4] is constructed from left-invariant currents $J^A = (g^{-1}\partial g)^A$ where $g(x,\theta,\widehat{\theta})$ takes values in the coset $PSU(2,2|4)/(SO(4,1) \times SO(5))$, $A = ([ab], m, \alpha, \widehat{\alpha})$ ranges over the 30 bosonic and

[1] In string theory, one usually assumes that any BRST-invariant operator of zero ghost-number can be put into Siegel gauge by adding an appropriate BRST-trivial operator. Siegel gauge implies that the operator commutes with the zero mode of the b ghost, so BRST-invariant operators in Siegel gauge commute with the hamiltonian $H = \{Q, b_0\}$. In the pure spinor formalism, there are no operators of negative ghost number, so there are no BRST-trivial operators of zero ghost number and there is no natural b ghost. It therefore appears that Siegel gauge is automatically imposed on ghost-number zero operators in the pure spinor formalism, implying that BRST-invariant charges of ghost-number zero necessarily commute with the hamiltonian.

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32 fermionic elements in the Lie algebra of PSU$(2,2|4)$, $[ab]$ labels the SO$(4,1)\times$ SO(5) "Lorentz" generators, $m=0$ to 9 labels the "translation" generators, and $\alpha,\widehat{\alpha}=1$ to 16 label the fermionic "supersymmetry" generators. The action in the pure spinor formalism also involves left and right-moving bosonic ghosts, $(\lambda^\alpha, w_\alpha)$ and $(\widehat{\lambda}^{\widehat{\alpha}}, \widehat{w}_{\widehat{\alpha}})$, which satisfy the pure spinor constraints $\lambda\gamma^m\lambda = \widehat{\lambda}\gamma^m\widehat{\lambda}=0$. Because of the pure spinor constraints, w_α and $\widehat{w}_{\widehat{\alpha}}$ can only appear in combinations which are invariant under $\delta w_\alpha = \xi^m(\gamma_m\lambda)_\alpha$ and $\delta\widehat{w}_{\widehat{\alpha}} = \widehat{\xi}^m(\gamma_m\widehat{\lambda})_{\widehat{\alpha}}$. These pure spinor ghosts couple to the $AdS_5\times S^5$ spin connection $J^{[ab]}$ in the worldsheet action through their Lorentz currents $N_{ab}=\frac{1}{2}w\gamma_{ab}\lambda$ and $\widehat{N}_{ab}=\frac{1}{2}\widehat{w}\gamma_{ab}\widehat{\lambda}$.

Using the notation defined below, the classical worldsheet action is

$$S_0 = \left\langle \frac{1}{2}J_2\overline{J}_2 + \frac{3}{4}J_3\overline{J}_1 + \frac{1}{4}J_1\overline{J}_3 + w\overline{\nabla}\lambda + \widehat{w}\nabla\widehat{\lambda} - N\widehat{N}\right\rangle \tag{2.1}$$

$$= \Bigg\langle \frac{1}{2}(J_2\overline{J}_2 + J_3\overline{J}_1 + J_1\overline{J}_3) + \frac{1}{4}(J_3\overline{J}_1 - J_1\overline{J}_3) + + (w\overline{\partial}\lambda + \widehat{w}\partial\widehat{\lambda} + N\overline{J}_0 + \widehat{N}J_0 - N\widehat{N})\Bigg\rangle\,, \tag{2.2}$$

where

$$\begin{aligned}
J_0 &= (g^{-1}\partial g)^{[ab]}T_{[ab]}\,, \quad J_1 = (g^{-1}\partial g)^{\alpha}T_\alpha\,, \quad J_2 = (g^{-1}\partial g)^m T_m\,, \quad J_3 = (g^{-1}\partial g)^{\widehat{\alpha}}T_{\widehat{\alpha}}\,,\\
w &= w_\alpha T_{\widehat{\alpha}}\delta^{\alpha\widehat{\alpha}}\,, \qquad \lambda = \lambda^\alpha T_\alpha\,, \qquad N = -\{w,\lambda\}\,,\\
\overline{J}_0 &= (g^{-1}\overline{\partial} g)^{[ab]}T_{[ab]}\,, \quad \overline{J}_1 = (g^{-1}\overline{\partial} g)^{\alpha}T_\alpha\,, \quad \overline{J}_2 = (g^{-1}\overline{\partial} g)^m T_m\,, \quad \overline{J}_3 = (g^{-1}\overline{\partial} g)^{\widehat{\alpha}}T_{\widehat{\alpha}}\,,\\
\widehat{w} &= \widehat{w}_{\widehat{\alpha}} T_{\alpha}\delta^{\alpha\widehat{\alpha}}\,, \qquad \widehat{\lambda} = \widehat{\lambda}^{\widehat{\alpha}} T_{\widehat{\alpha}}\,, \qquad \widehat{N} = -\{\widehat{w},\widehat{\lambda}\}\,,\\
\nabla Y &= \partial Y + [J_0, Y]\,, \qquad \overline{\nabla}Y = \overline{\partial}Y + [\overline{J}_0, Y]\,,
\end{aligned} \tag{2.3}$$

$\delta_{\alpha\widehat{\beta}} = (\gamma^{01234})_{\alpha\widehat{\beta}}$, $\langle w\overline{\partial}\lambda + \widehat{w}\partial\widehat{\lambda}\rangle$ is the action in a flat background for the pure spinors, T_A are the PSU$(2,2|4)$ Lie algebra generators, and $\langle\ \ \rangle$ denotes a super-trace over the PSU$(2,2|4)$ matrices and integration over the two-dimensional worldsheet, e.g. $\langle J_2\overline{J}_2\rangle = \int d^2z (g^{-1}\partial g)^m (g^{-1}\overline{\partial}g)^n STr(T_mT_n)$. Note that

$$\{T_\alpha, T_\beta\} = \gamma^m_{\alpha\beta}T_m\,, \qquad \{T_{\widehat{\alpha}}, T_{\widehat{\beta}}\} = \gamma^m_{\widehat{\alpha}\widehat{\beta}}T_m\,, \qquad \{T_\alpha, T_{\widehat{\beta}}\} = \left(\frac{1}{2}\gamma^{[ab]}\gamma^{01234}\right)_{\alpha\widehat{\beta}} T_{[ab]}\,, \tag{2.4}$$

and

$$\mathrm{Str}(T_{[ab]}T_{[cd]}) = \delta_{a[c}\delta_{d]b}\,, \quad \mathrm{Str}(T_mT_n) = \eta_{mn}\,, \quad \mathrm{Str}(T_\alpha T_{\widehat{\beta}}) = -\,\mathrm{Str}(T_{\widehat{\beta}}T_\alpha) = \delta_{\alpha\widehat{\beta}}\,.$$

The action of (2.1) is manifestly invariant under global PSU$(2,2|4)$ transformations which transform $g(x,\theta,\widehat{\theta})$ by left multiplication as $\delta g = (\Sigma^A T_A)g$ and is also manifestly invariant under local SO$(4,1)\times$ SO(5) gauge transformations which transform $g(x,\theta,\widehat{\theta})$ by right multiplication as $\delta_\Lambda g = g\Lambda$ and transform the pure spinors as

$$\delta_\Lambda\lambda = [\lambda,\Lambda], \quad \delta_\Lambda\widehat{\lambda} = [\widehat{\lambda},\Lambda], \quad \delta_\Lambda w = [w,\Lambda], \quad \delta_\Lambda\widehat{w} = [\widehat{w},\Lambda]\,,$$

where $\Lambda = \Lambda^{[ab]}T_{[ab]}$.

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Under classical BRST transformations generated by

$$\epsilon Q = \epsilon \int d\sigma \, \mathrm{STr} \left(\lambda J_3 + \widehat{\lambda} \overline{J}_1 \right),$$

where ϵ is a constant anticommuting parameter, $g(x, \theta, \widehat{\theta})$ transforms by right-multiplication as

$$\epsilon Q(g) = g(\epsilon \lambda + \epsilon \widehat{\lambda}) \tag{2.5}$$

and the pure spinors transform as

$$\epsilon Q(w) = -J_3 \epsilon \,, \qquad \epsilon Q(\widehat{w}) = -\overline{J}_1 \epsilon \,, \qquad \epsilon Q(\lambda) = \epsilon Q(\widehat{\lambda}) = 0 \,,$$

which implies that

$$\epsilon Q(N) = [J_3, \epsilon \lambda] \,, \qquad \epsilon Q(\widehat{N}) = [\overline{J}_1, \epsilon \widehat{\lambda}] \,.$$

The left-invariant currents of (2.3) transform under (2.5) as

$$\begin{aligned} \epsilon Q(J_j) &= \delta_{j+3,0} \, \partial(\epsilon \lambda) + [J_{j+3}, \epsilon \lambda] + \delta_{j+1,0} \, \partial \left(\epsilon \widehat{\lambda} \right) + \left[J_{j+1}, \epsilon \widehat{\lambda} \right], \\ \epsilon Q(\overline{J}_j) &= \delta_{j+3,0} \, \overline{\partial}(\epsilon \lambda) + [\overline{J}_{j+3}, \epsilon \lambda] + \delta_{j+1,0} \, \overline{\partial} \left(\epsilon \widehat{\lambda} \right) + \left[\overline{J}_{j+1}, \epsilon \widehat{\lambda} \right], \end{aligned}$$

where j is defined modulo 4, i.e. $J_j \equiv J_{j+4}$.

One can easily verify that S_0 is the unique $\mathrm{PSU}(2,2|4)$-invariant expression which is BRST invariant under (2.5). To verify this, note that the first term in (2.2) transforms under (2.5) to

$$\frac{1}{2} \langle J_3 \overline{\nabla}(\epsilon \lambda) + \overline{J}_3 \nabla(\epsilon \lambda) + J_1 \overline{\nabla}(\epsilon \widehat{\lambda}) + \overline{J}_1 \nabla(\epsilon \widehat{\lambda}) \rangle \,.$$

Using the Maurer-Cartan equations

$$\nabla \overline{J}_3 - \overline{\nabla} J_3 = -[J_1, \overline{J}_2] - [J_2, \overline{J}_1] \,, \qquad \nabla \overline{J}_1 - \overline{\nabla} J_1 = -[J_3, \overline{J}_2] - [J_2, \overline{J}_3] \,, \tag{2.6}$$

the second term in (2.2) transforms under (2.5) to

$$\frac{1}{2} \langle J_3 \overline{\nabla}(\epsilon \lambda) - \overline{J}_3 \nabla(\epsilon \lambda) - J_1 \overline{\nabla}(\epsilon \widehat{\lambda}) + \overline{J}_1 \nabla(\epsilon \widehat{\lambda}) \rangle \,.$$

And the last term in (2.2) transforms under (2.5) to

$$\langle -J_3 \overline{\nabla}(\epsilon \lambda) - \overline{J}_1 \nabla(\epsilon \widehat{\lambda}) \rangle \,.$$

3. $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ gauge invariance

Using the classical BRST transformation of (2.5) and $\{\lambda, \lambda\} = \{\widehat{\lambda}, \widehat{\lambda}\} = 0$ from the pure spinor constraints, one finds that

$$\begin{aligned} Q^2(g) &= -g \left\{ \lambda, \widehat{\lambda} \right\}, \\ Q^2(N) &= -\left[N, \{\lambda, \widehat{\lambda}\} \right] - \left\{ \lambda, \nabla \widehat{\lambda} - [N, \widehat{\lambda}] \right\}, \\ Q^2(\widehat{N}) &= -\left[\widehat{N}, \{\lambda, \widehat{\lambda}\} \right] - \left\{ \widehat{\lambda}, \overline{\nabla} \lambda - [\widehat{N}, \lambda] \right\}. \end{aligned} \tag{3.1}$$

Furthermore, $[\lambda, \{\lambda, \widehat{\lambda}\}] = [\widehat{\lambda}, \{\lambda, \widehat{\lambda}\}] = 0$ implies that

$$Q^2(\lambda) = 0 = -\left[\lambda, \{\lambda, \widehat{\lambda}\}\right], \qquad Q^2(\widehat{\lambda}) = 0 = -\left[\widehat{\lambda}, \{\lambda, \widehat{\lambda}\}\right]. \tag{3.2}$$

So up to an $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ gauge transformation parameterized by

$$\{\lambda, \widehat{\lambda}\} = \lambda^{\alpha} \widehat{\lambda}^{\widehat{\beta}} \left(\frac{1}{2} \gamma^{[ab]} \gamma^{01234}\right)_{\alpha \widehat{\beta}} T_{[ab]}, \tag{3.3}$$

and up to the classical equations of motion

$$\nabla \widehat{\lambda} - \left[N, \widehat{\lambda}\right] = 0 \qquad \text{and} \qquad \overline{\nabla} \lambda - \left[\widehat{N}, \lambda\right] = 0, \tag{3.4}$$

Q is nilpotent. Since the classical action of (2.1) is invariant under $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ gauge transformations, Q is therefore a consistent BRST transformation at the classical level.

It will now be argued that after adding a local counterterm, the quantum effective action remains invariant under $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ gauge transformations. This is essential for consistency of the BRST transformation at the quantum level. To prove that such a local counterterm can always be found, note that the $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ gauge transformation of the quantum effective action, $\delta_\Lambda S_q$, must be a local operator since any quantum anomaly comes from a short-distance regulator. Furthermore, since global $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ invariance is manifest, $\delta_\Lambda S_q$ must vanish when the $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ gauge parameter $\Lambda = \Lambda^{[ab]} T_{[ab]}$ is constant. Therefore,

$$\delta_\Lambda S_q = \int d^2 z \left(f_{[ab]} \overline{\partial} \Lambda^{[ab]} + \overline{f}_{[ab]} \partial \Lambda^{[ab]}\right), \tag{3.5}$$

where $f_{[ab]}$ and $\overline{f}_{[ab]}$ are some operators which carry (left, right) conformal weight $(1,0)$ and $(0,1)$ respectively. The only candidates for $f_{[ab]}$ and $\overline{f}_{[ab]}$ are $(N_{[ab]}, J_{[ab]})$ and $(\overline{N}_{[ab]}, \overline{J}_{[ab]})$, so

$$\delta_\Lambda S_q = \langle c_1 N \overline{\partial} \Lambda + \overline{c}_1 \widehat{N} \partial \Lambda + c_2 J_0 \overline{\partial} \Lambda + \overline{c}_2 \overline{J}_0 \partial \Lambda \rangle \tag{3.6}$$

for some constants $(c_1, \overline{c}_1, c_2, \overline{c}_2)$. By adding the local counterterm

$$S_c = -\left\langle c_1 N \overline{J}_0 + \overline{c}_1 \widehat{N} J_0 + \frac{1}{2}(c_2 + \overline{c}_2) J_0 \overline{J}_0 \right\rangle, \tag{3.7}$$

one can cancel most of the variation to obtain

$$\delta_\Lambda (S_q + S_c) = \frac{1}{2}(c_2 - \overline{c}_2) \left\langle J_0 \overline{\partial} \Lambda - \overline{J}_0 \partial \Lambda \right\rangle, \tag{3.8}$$

which is the standard parity-violating anomalous variation in two dimensions.

Although the worldsheet action of (2.1) is not invariant under a parity transformation which exchanges z with $\overline{z}$, the action is invariant under a transformation which simultaneously exchanges z with $\overline{z}$, λ with $\widehat{\lambda}$, w with $\widehat{w}$, and θ with $\widehat{\theta}$. This implies that $c_1 = \overline{c}_1$ and $c_2 = \overline{c}_2$ in $\delta_\Lambda S_q$ in (3.6). So the anomalous variation of (3.8) vanishes, implying that $\delta_\Lambda (S_q + S_c) = 0$.

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4. Quantum BRST invariance

In this section, trivial BRST cohomology at ghost-number +1 will be used to prove that the BRST transformation of the quantum effective action S_q can be cancelled by adding a local counterterm. Since the BRST transformation of (2.5) commutes with SO(4,1)×SO(5) gauge transformations and since $\delta_\Lambda S_q = 0$ after adding the counterterm of the previous section, the BRST transformation of S_q satisifes $\delta_\Lambda Q(S_q) = 0$. Furthermore, the BRST variation of the quantum effective action must be a local operator since quantum anomalies come from a short-distance regulator.

So $Q(S_q)$ is an $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ gauge-invariant local operator of ghost-number +1, which implies it can be written as

$$\epsilon Q(S_q) = \langle a_1 \overline{J}_2[J_3, \epsilon\widehat{\lambda}] + \overline{a}_1 J_2[\overline{J}_1, \epsilon\lambda] + a_2\overline{J}_2[J_1,\epsilon\lambda] + \overline{a}_2 J_2[\overline{J}_3, \epsilon\widehat{\lambda}] + \\ + a_3 J_3[\widehat{N}, \epsilon\lambda] + \overline{a}_3\overline{J}_1[N, \epsilon\widehat{\lambda}] + a_4 J_3\overline{\nabla}(\epsilon\lambda) + \overline{a}_4\overline{J}_1\nabla(\epsilon\widehat{\lambda})\rangle\,, \tag{4.1}$$

where a_j and $\overline{a}_j$ are some constants.[2] Note that $[\widehat{N}, \epsilon\widehat{\lambda}] = [N, \epsilon\lambda] = 0$ because of the pure spinor constraint and that terms such as $\langle \overline{J}_3 \nabla(\epsilon\lambda)\rangle$ can be related to terms in (4.1) by integrating by parts and using the Maurer-Cartan equation $\nabla\overline{J}_3 - \overline{\nabla} J_3 = [\overline{J}_1, J_2] + [\overline{J}_2, J_1]$.

Since Q is nilpotent on $\mathrm{SO}(4,1) \times \mathrm{SO}(5)$ gauge-invariant operators up to the equations of (3.4), $Q^2(S_q)$ must be proportional to the equations of (3.4). Using the BRST transformations of (2.5), this implies that the coefficients in (4.1) must satisfy[3]

$$a_1 = \overline{a}_1\,, \qquad a_2 = \overline{a}_2\,, \qquad a_3 + a_4 = \overline{a}_3 + \overline{a}_4\,. \tag{4.2}$$

It will now be shown that whenever the restriction of (4.2) is satisfied, $Q(S_q)$ can be written as the BRST variation of a local counterterm. In other words, the BRST cohomology of local ghost-number +1 operators is trivial. Using the BRST transformations of (2.5), one finds that the local counterterm S_c which satisfies $Q(S_c) = -Q(S_q)$ is

$$S_c = \langle -a_2\overline{J}_2 J_2 + (a_1 - a_2)\overline{J}_1 J_3 + (a_3 - \overline{a}_4 + a_2 - a_1) N\widehat{N} + \\ + (a_4 + a_1 - a_2) w\overline{\nabla}\lambda + (\overline{a}_4 + a_1 - a_2)\widehat{w}\nabla\widehat{\lambda}\rangle\,. \tag{4.3}$$

So $Q(S_q + S_c) = 0$, implying that the quantum effective action $S_{\text{eff}} = S_q + S_c$ is invariant under BRST transformations.

Using the algebraic renormalization method of [9], this proof of quantum BRST invariance can be extended by induction to all orders in perturbation theory.[4] For example, suppose the quantum effective action is BRST invariant up to order h^{n-1}, i.e.

[2] Terms such as $\langle \overline{J}_2\{J_3, \epsilon\widehat{\lambda}\}\rangle$ do not need to be considered since the effective action (e.g., using the background field method) and BRST transformations only involve the structure constants f_{AB}^C and do not involve constants such as d_{AB}^C coming from anticommutators.

[3] I would like to thank Brenno Vallilo for discussions on this computation.

[4] Although the algebraic renormalization method of [9] uses the "gauge-invariant" BRST cohomology including antifields, the proof here uses the "gauge-fixed" BRST cohomology where antifields have been set to zero. As discussed in [14], the gauge-fixed cohomology is sufficient for proving quantum BRST invariance if quantum modifications to the gauge-fixed BRST operator can be defined such that nilpotence is preserved. This is possible if there are no conserved currents of ghost-number 2 which could deform Q^2.

$\widetilde{Q}(S_{\text{eff}}) = h^n\Omega + \mathcal{O}(h^{n+1})$ for some local Ω where $\widetilde{Q} = Q + Q_q$ and Q_q generates quantum corrections to the classical BRST transformations of (2.5) generated by Q. Since $Q\Omega = 0$, trivial cohomology at ghost-number $+1$ implies that $\widetilde{Q}(S_{\text{eff}} - h^n\Sigma) = \mathcal{O}(h^{n+1})$ where Σ is a local operator satisfying $Q\Sigma = \Omega$. So the quantum effective action $S_{\text{eff}} - h^n\Sigma$ is BRST invariant up to order h^n.

5. Quantum conformal invariance

To prove that the quantum effective action is conformally invariant, a trick shall be used which was previously used for the superstring in an $AdS_3 \times S^3$ Ramond-Ramond background [10]. The trick is to enlarge the PSU(2, 2|4) Lie algebra to a U(2, 2|4) Lie algebra. In other words, include two new bosonic generators, I and L, satisfying the commutation relations

$$\begin{aligned} &[L, T_\alpha] = \delta_\alpha^{\widehat{\beta}} T_{\widehat{\beta}}\,, \qquad [L, T_{\widehat{\alpha}}] = -\delta_{\widehat{\alpha}}^{\beta} T_\beta\,, \\ &\{T_\alpha, T_\beta\} = \gamma^m_{\alpha\beta} T_m + (\gamma^{01234})_{\alpha\beta} I\,, \qquad \{T_{\widehat{\alpha}}, T_{\widehat{\beta}}\} = \gamma^m_{\widehat{\alpha}\widehat{\beta}} T_m + (\gamma^{01234})_{\widehat{\alpha}\widehat{\beta}} I\,. \end{aligned} \tag{5.1}$$

So the U(2, 2|4) generators (I, L, T_A) satisfy the algebra

$$[L, T_A] = c_A^B T_B\,, \qquad [T_A, T_B\} = f_{AB}^C T_C + d_{AB} I\,, \qquad [I, T_A] = [I, L] = 0\,, \tag{5.2}$$

where f_{AB}^C are the PSU(2, 2|4) structure constants, $c_\alpha^{\widehat{\beta}} = \delta_\alpha^{\widehat{\beta}}$, $c_{\widehat{\alpha}}^{\beta} = -\delta_{\widehat{\alpha}}^{\beta}$, $d_{\alpha\beta} = \gamma^{01234}_{\alpha\beta}$ and $d_{\widehat{\alpha}\widehat{\beta}} = \gamma^{01234}_{\widehat{\alpha}\widehat{\beta}}$. Note that L acts as an outer automorphism of PSU(2, 2|4) and I acts as a central extension.

Now define left-invariant currents

$$K = h^{-1}\partial h\,, \qquad \overline{K} = h^{-1}\overline{\partial} h\,, \tag{5.3}$$

where $h(x, \theta, \widehat{\theta}, u, v)$ takes values in the coset U(2, 2|4)/(SO(4, 1) × SO(5)) and (u, v) are two additional bosonic variables which are not present in the coset PSU(2, 2|4)/(SO(4, 1) × SO(5)). It is convenient to parameterize

$$h(x, \theta, \widehat{\theta}, u, v) = \exp(uI + vL)\; g(x, \theta, \widehat{\theta})\,, \tag{5.4}$$

where $g(x, \theta, \widehat{\theta})$ takes values in PSU(2, 2|4)/(SO(4, 1) × SO(5)). So

$$h^{-1}\partial h = K_I + K_L + K_0 + K_1 + K_2 + K_3 \tag{5.5}$$

where

$$\begin{aligned} K_I &= (h^{-1}\partial h)^I I = (\partial u + (g^{-1}(\partial v\; L)g)^I + (g^{-1}\partial g)^I) I\,, \\ K_L &= (h^{-1}\partial h)^L L = (\partial v) L\,, \\ K_0 + K_1 + K_2 + K_3 &= (h^{-1}\partial h)^A T_A = ((g^{-1}(\partial v\; L)g)^A + (g^{-1}\partial g)^A) T_A\,. \end{aligned}$$

A counter-example discussed in [14] is the conserved current $j_\mu = C\partial_\mu C$ in Maxwell theory where C is the fermionic ghost whose equation of motion in Lorentz gauge is $\partial_\mu\partial^\mu C = 0$. Fortunately, one can easily check that there are no conserved currents of ghost-number 2 for the action of (2.1), so the gauge-fixed BRST cohomology is sufficient for proving quantum BRST invariance.

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Under the BRST transformation $\epsilon Q'(h) = h(\epsilon\lambda + \epsilon\widehat{\lambda})$, the left-invariant currents transform as

$$\begin{aligned} \epsilon Q'(K_I) &= [K_3, \epsilon\widehat{\lambda}] + [K_1, \epsilon\lambda]\,, \qquad \epsilon Q'(K_L) = 0\,, \\ \epsilon Q'(K_j) &= \delta_{j+3,0}(\epsilon\partial\lambda + [K_L, \epsilon\widehat{\lambda}]) + [K_{j+3}, \epsilon\lambda] + \delta_{j+1,0}(\epsilon\overline{\partial}\widehat{\lambda} + [\overline{K}_L, \epsilon\lambda]) + [K_{j+1}, \epsilon\widehat{\lambda}]\,, \end{aligned} \tag{5.6}$$

where j is defined modulo 4, i.e. $K_j \equiv K_{j+4}$.

Now consider the classical worldsheet action

$$S' = \frac{1}{2}\int d^2z\, \mathrm{Str}\left(K_I \overline{K}_L + \overline{K}_I K_L\right) + S'_0\,, \tag{5.7}$$

where S'_0 is the classical action of (2.1) with J_A replaced by K_A. Note that S' is manifestly invariant under global $\mathrm{U}(2,2|4)$ transformations which transform h by left multiplication, and differs from S_0 because of its dependence on the two additional bosons u and v. Using the $\mathrm{U}(2,2|4)$ Maurer-Cartan equations,

$$\begin{aligned} \nabla\overline{K}_3 - \overline{\nabla}K_3 &= -[K_1, \overline{K}_2] - [K_2, \overline{K}_1] - [K_L, \overline{K}_1] - [K_1, \overline{K}_L]\,, \\ \nabla\overline{K}_1 - \overline{\nabla}K_1 &= -[K_3, \overline{K}_2] - [K_2, \overline{K}_3] - [K_L, \overline{K}_3] - [K_3, \overline{K}_L]\,, \end{aligned} \tag{5.8}$$

one can check that S' is invariant under the BRST transformations of (5.6). Furthermore, one can repeat the arguments of sections 3 and 4 to show that the BRST transformation of the quantum effective action S'_q can be cancelled by adding a local counterterm S'_c to obtain a BRST-invariant quantum action $S'_{\mathrm{eff}} = S'_q + S'_c$.

From the definitions in (5.5) for the left-invariant currents,

$$S' = S_0 + \int d^2z(\partial u\overline{\partial}v + j(x,\theta,\widehat{\theta})\overline{\partial}v + \overline{j}(x,\theta,\widehat{\theta})\partial v + k(x,\theta,\widehat{\theta})\partial v\overline{\partial}v)\,,$$

where S_0, j, $\overline{j}$ and k are independent of u and v. Since there are no terms in S' which are quadratic in u, there is no $\langle vv\rangle$ propagator in the Feynman rules for the quantum effective action. So v-independent terms in the quantum effective action S'_{eff} are the same as in the original $\mathrm{PSU}(2,2|4)$-invariant quantum effective action S_{eff} of section 4. It will now be proven that S'_{eff} is conformally invariant, which immediately implies that S_{eff} is conformally invariant since $S'_{\mathrm{eff}}|_{v=0} = S_{\mathrm{eff}}$.

To prove that S'_{eff} is conformally invariant, first note that cohomology arguments imply that quantum conformal transformations can be defined to commute with quantum BRST transformations.[5] So if δ_C denotes the quantum conformal transformation, $Q'(\delta_C S'_{\mathrm{eff}}) = 0$. So the conformal transformation of S'_{eff} must be BRST invariant, and must be local since it comes from a short-distance regulator. But one can easily verify that the only $\mathrm{U}(2,2|4)$-invariant BRST-invariant local operator of ghost-number zero is the classical action S'

[5]To prove this, suppose that $[\widetilde{Q}, \delta_C] = h^n\delta' + \mathcal{O}(h^{n+1})$ where $\widetilde{Q} = Q + Q_q$, Q_q generates quantum corrections to the classical BRST transformations generated by Q, δ_C is the conformal transformation to order h^{n-1}, and δ' is some local transformation carrying +1 ghost number. Then $\{Q, \delta'\} = 0$ implies that $\delta' = -[Q, \delta_q]$ for some δ_q because of trivial BRST cohomology for local charges of +1 ghost number. So $\delta_C + h^n\delta_q$ can be defined as the conformal transformation to order h^n, and satisfies $[\widetilde{Q}, \delta_C + h^n\delta_q] = \mathcal{O}(h^{n+1})$.

of (5.7). So $\delta_C S'_{\rm eff}$ must be proportional to S'. But the term $\int d^2z\partial u\overline{\partial}v$ in S' cannot receive quantum corrections since S' contains no terms quadratic in u. So the term $\int d^2z\partial u\overline{\partial}v$ cannot appear in $\delta_C S'_{\rm eff}$, which implies that $\delta_C S'_{\rm eff} = 0$. Note that as in the proof of quantum BRST invariance in section 4, this proof of quantum conformal invariance is valid to all orders in perturbation theory.[6]

6. Non-local BRST-invariant charges

As discussed in [7], the existence of non-local BRST-invariant charges in string theory is related to the triviality of a certain BRST cohomology class. To understand this relation, consider the non-local integrated operator

$$k^C = f^C_{AB} \int_{-\infty}^{\infty} d\sigma\ j^A(\sigma)\ \int_{-\infty}^{\sigma} d\sigma'\ j^B(\sigma')\,, \tag{6.1}$$

where $\int_{-\infty}^{\infty} d\sigma j^A(\sigma)$ are the Noether charges for the global symmetry algebra and f^C_{AB} are the structure constants. Since the Noether charges are BRST-invariant, $Q(j^A(\sigma)) = \partial_\sigma h^A(\sigma)$ for some $h^A(\sigma)$ of $+1$ ghost-number, which implies that $Q(k^C) = -2f^C_{AB}\int_{-\infty}^{\infty} d\sigma\ h^A(\sigma)j^B(\sigma)$ is a local integrated operator of $+1$ ghost-number.

Whenever $Q(k^C)$ can be written as the BRST variation of a local integrated operator, i.e. whenever $Q(k^C) = Q(\int_{-\infty}^{\infty} d\sigma\Sigma^C(\sigma))$ for some local $\Sigma^C(\sigma)$, one can construct the non-local BRST-invariant charge

$$q^C = f^C_{AB} \int_{-\infty}^{\infty} d\sigma\ j^A(\sigma)\ \int_{-\infty}^{\sigma} d\sigma'\ j^B(\sigma')\ - \int_{-\infty}^{\infty} d\sigma\Sigma^C(\sigma)\,. \tag{6.2}$$

Furthermore, by repeatedly commuting q^C with q^D, one generates an infinite set of non-local BRST-invariant charges.

So if the BRST cohomology is trivial for local integrated operators of $+1$ ghost-number transforming in the adjoint representation, one can construct an infinite set of non-local BRST-invariant charges. Furthermore, one can use arguments similar to those of section 4 to prove that this construction is valid at the quantum level to all orders in perturbation theory. For example, suppose that a non-local BRST-invariant charge q^A has been constructed to order h^{n-1}, i.e. $\widetilde{Q}(q^C) = h^n\Omega^C + \mathcal{O}(h^{n+1})$ where Ω^C is some integrated operator of ghost-number $+1$, $\widetilde{Q} = Q + Q_q$, and Q_q generates quantum corrections to the classical BRST transformations of (2.5) generated by Q. Like other types of quantum anomalies, Ω^C must be a local integrated operator since it comes from a short-distance regulator in the operator product expansion $j^A(\sigma)j^B(\sigma')$ [15]. So trivial cohomology implies that there exists a local operator $\Sigma^C(\sigma)$ such that $\Omega^C = Q(\int_{-\infty}^{\infty} d\sigma\Sigma^C(\sigma))$. Therefore, $q^C - h^n\int_{-\infty}^{\infty} d\sigma\Sigma^C(\sigma)$ is BRST-invariant to order h^n.

[6]At zeroth order in perturbation theory, the model is conformally invariant for the same reason as in a flat background. Namely, the contribution to the central charge from the 10 x's and 32 θ's and $\widehat{\theta}$'s of the $AdS_5\times S^5$ background cancels the contribution from the 22 λ and $\widehat{\lambda}$ pure spinors. The above arguments imply that this cancellation holds to all orders in perturbation theory.

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To verify that the relevant cohomology class is trivial for the superstring in an $AdS_5\times S^5$ background, it will be useful to recall that for every integrated operator of ghost-number +1 in the BRST cohomology, there exists a corresponding unintegrated operator of ghost-number +2 and zero conformal weight in the BRST cohomology. This is easy to prove since $Q(\int d\sigma W(\sigma))=0$ implies that $Q(W(\sigma))=\partial_\sigma V(\sigma)$ where $V(\sigma)$ is a BRST-invariant operator of zero conformal weight. And if V is BRST-trivial, i.e. if $V=Q\Lambda$ for some Λ, then $Q(W-\partial_\sigma\Lambda)=0$. Since the BRST cohomology is trivial for unintegrated operators of nonzero conformal weight, $W-\partial_\sigma\Lambda=Q\Sigma$ for some Σ. So

$$\int d\sigma W(\sigma)=\int d\sigma(Q\Sigma(\sigma)+\partial_\sigma\Lambda(\sigma))=Q\left(\int d\sigma\Sigma(\sigma)\right), \tag{6.3}$$

which implies that $\int d\sigma W(\sigma)$ is BRST-trivial [16].

At ghost-number +2, the only unintegrated operators of zero conformal weight which transform in the adjoint representation of the global PSU(2, 2|4) algebra are

$$V_1=g\,(\lambda^\alpha T_\alpha)\left(\widehat{\lambda}^{\widehat{\beta}}T_{\widehat{\beta}}\right)\,g^{-1}\qquad\text{and}\qquad V_2=g\left(\widehat{\lambda}^{\widehat{\alpha}}T_{\widehat{\alpha}}\right)\left(\lambda^\beta T_\beta\right)\,g^{-1}\,, \tag{6.4}$$

where $g(x,\theta,\widehat{\theta})$ transforms by left multiplication as $\delta g(x,\theta,\widehat{\theta})=\Sigma g(x,\theta,\widehat{\theta})$ under the global PSU(2, 2|4) transformation parameterized by $\Sigma=\Sigma^A T_A$. One can easily verify that $Q(V_1-V_2)\neq 0$ and that $V_1+V_2=Q\Omega$ where

$$\Omega=\frac{1}{2}g\,\left(\lambda^\alpha T_\alpha+\widehat{\lambda}^{\widehat{\alpha}}T_{\widehat{\alpha}}\right)\,g^{-1}\,. \tag{6.5}$$

So the cohomology is trivial, which implies the existence of an infinite set of non-local BRST-invariant charges at the quantum level.

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